

Arithmetic on algebraic fractions

Donald Campbell

Summary

Algebraic fractions extend numerical fractions to expressions that include variables. This guide explains how to add, subtract, multiply and divide algebraic fractions, including how to simplify fractions within fractions. You will see how techniques for arithmetic on numerical fractions still apply, provided you factorize carefully and keep track of any restricted values where the denominator would be zero.

Before reading this guide, it is recommended that you read [Guide: Introduction to numerical fractions](#), [Guide: Arithmetic on numerical fractions](#), [Guide: Introduction to algebraic fractions](#) and [Guide: Factorization](#).

In [Guide: Introduction to algebraic fractions](#), you learned how to:

- identify restrictions on the denominator for values that make it zero
- rewrite algebraic fractions in equivalent forms
- simplify algebraic fractions by factorizing and cancelling common factors

In this guide, you will see how the arithmetic rules from [Guide: Arithmetic on numerical fractions](#) still apply to algebraic fractions. You will learn how to add, subtract, multiply and divide algebraic fractions, and simplify compound fractions (fractions within fractions), while keeping track of any restricted values.

Adding and subtracting algebraic fractions

When adding or subtracting algebraic fractions, the key idea is that the denominators of both fractions must be the same. The denominator tells you the size of each part, and you can only combine parts that are of equal size.

As with numerical fractions, **you cannot add directly if the denominators are different**. Instead, you first rewrite the fractions as equivalent fractions with a common denominator, then add or subtract the numerators. Thankfully, if the denominators are the same, then you can proceed straight away:

i Adding and subtracting fractions with the same denominator

To add or subtract fractions, the denominators of both fractions **must be the same**. If the fractions already have the same denominator, add or subtract the numerators and keep the same denominator.

- If the fractions have different denominators, you could do the following:

Step 1: Multiply the denominators together.

Step 2: Rewrite the two terms in the addition/subtraction as equivalent fractions so that the two new fractions have the same denominator as found in Step 1.

Step 3: Add or subtract the numerators and keep the same denominator.

Step 4: Cancel any common factors if needed.

For more about writing equivalent algebraic fractions or cancelling common factors, please see [Guide: Introduction to algebraic fractions](#).

i Example 1

Find $\frac{2}{x} + \frac{5}{x}$.

Both fractions already have denominator equal to x , so you can add the numerators and keep the same denominator.

$$\frac{2}{x} + \frac{5}{x} = \frac{2+5}{x} = \frac{7}{x} \quad \text{if } x \neq 0$$

There are no common factors other than 1, so the fraction is already in its simplest form.

When the denominators are different, you need to find a common denominator. For algebraic fractions, this usually means multiplying the denominators together or using their lowest common multiple, similar to what you did for numerical fractions.

i Adding and subtracting fractions with different denominators

To add or subtract fractions, the denominators of both fractions **must be the same**. If the fractions have different denominators, you could do the following:

1. Multiply the denominators together.
2. Rewrite the two terms in the addition/subtraction as equivalent fractions so that the two new fractions have the same denominator as found in Step 1.
3. Add or subtract the numerators and keep the same denominator.

4. Cancel any common factors if needed.

i Example 2

Find $\frac{1}{x} + \frac{1}{x+1}$.

Both fractions have different denominators. The denominators are x and $x+1$.

For step 1, a common denominator for the fractions is found by multiplying the denominators together:

$$x(x+1)$$

In step 2, you can write both fractions as equivalent fractions with this common denominator, by multiplying the numerator and denominator by the same term. Remember that what you do to the denominator must also be done to the numerator to keep the fractions equivalent. Here, multiplying $1/x$ by $x+1$ on both top and bottom gives an equivalent fraction with denominator $x(x+1)$ as found in step 1.

So

$$\frac{1}{x} = \frac{x+1}{x(x+1)}$$

Similarly, multiplying $1/x+1$ by x on both top and bottom gives an equivalent fraction with denominator $x(x+1)$ as found in step 1.

$$\frac{1}{x+1} = \frac{x}{x(x+1)}$$

Now the denominators are the same, you can add the numerators:

$$\begin{aligned}\frac{1}{x} + \frac{1}{x+1} &= \frac{x+1}{x(x+1)} + \frac{x}{x(x+1)} \\ &= \frac{x+1+x}{x(x+1)} \\ &= \frac{2x+1}{x(x+1)} \quad \text{if } x \neq 0 \text{ and } x \neq -1\end{aligned}$$

This fraction cannot be simplified further as there are no common factors other than 1. So the fraction is in its simplest form and this is your final answer.

💡 Tip

This process can be done more quickly by using a technique called **cross-multiplication**. For two fractions A/B and C/D , this can be done by

- multiplying B and D together to get BD to get your denominator of your added

fraction

- multiplying the A and D to get AD and multiplying B and C to get BC ; then add AD and BC to get the numerator of your added fraction.

In short:

$$\frac{A}{B} + \frac{C}{D} = \frac{AD + BC}{BD}.$$

The reason why this is called 'cross-multiplication' is because you are 'crossing' the denominators of each added term to the numerator of the other.

Here's an example of cross-multiplication. This working is probably closer to what you would expect to do if you are working on these problems on your own.

i Example 3

Find $\frac{1}{x} + \frac{1}{x+1}$.

Here, the denominators are x and $x+1$, so multiply together to get $x(x+1)$. This is the denominator of your final answer.

Multiply the numerator of the first fraction 1 by the denominator of the second $x+1$ to get $x+1$. Similarly, multiply the numerator of the second fraction 1 by the denominator of the first x to get x . Adding these together gives $x+1+x = 2x+1$.

In short:

$$\begin{aligned}\frac{1}{x} + \frac{1}{x+1} &= \frac{(x+1) + x}{x(x+1)} \\ &= \frac{2x+1}{x(x+1)} \quad \text{if } x \neq 0 \text{ and } x \neq -1\end{aligned}$$

You know from Example 2 that this fraction is in its simplest form.

Typically with algebraic fractions, it's better to multiply all denominators together rather than try to find a 'lowest common denominator'. This is because you can always factorise and cancel at the end. Here's an example of this in action, using the cross multiplication method.

Example 4

Find $\frac{4}{t} - \frac{1}{t(t-1)}$.

Here, the denominators are t and $t(t-1)$, so multiply together to get $t^2(t-1)$. This is the denominator of your final answer.

Multiply the numerator of the first fraction 4 by the denominator of the second $t(t-1)$ to get $4t(t-1)$. Similarly, multiply the numerator of the second fraction 1 by the denominator of the first t to get t . Subtracting these gives $4t(t-1) - t$. (It is very tempting to expand the brackets at this stage; but don't!)

So in short:

$$\frac{4}{t} - \frac{1}{t(t-1)} = \frac{4t(t-1) - t}{t^2(t-1)}$$

Here, you can notice that there is a common factor of t on top and bottom. Unpacking and re-factorising the numerator gives

$$4t(t-1) - t = 4t^2 - 4t - t = 4t^2 - 5t = t(4t-5).$$

You can substitute the working in, then cancel the factor of t from top and bottom to get:

$$\begin{aligned} \frac{4}{t} - \frac{1}{t(t-1)} &= \frac{4t(t-1) - t}{t^2(t-1)} = \frac{t(4t-5)}{t^2(t-1)} \\ &= \frac{4t-5}{t(t-1)} \quad \text{if } t \neq 0 \text{ and } t \neq 1 \end{aligned}$$

and this is your final answer.

A good rule to follow when adding/subtracting algebraic fractions is the following:

Warning

Don't expand brackets unless you really have to!

A sub-rule of this is that **it's also better to wait until the very end of your working to do so**. This is to reduce the chances of a factorization error in your working, where any factorization is not explicitly needed.

Multiplying algebraic fractions

When you multiply two fractions, you are finding a fraction of another fraction. The method used for numerical fractions still works for algebraic fractions, in that you multiply the numer-

ators together and you multiply the denominators together.

i Example 5

For instance:

$$\frac{1}{x} \cdot \frac{y}{2} = \frac{1 \cdot y}{x \cdot 2} = \frac{y}{2x}$$

⚠ Warning

You can notice that the alternative notation $\cdot$ is used for multiplication here, instead of $\times$. This is because $\times$ and x look too similar!

For more complicated expressions, which could involve multiple factors or terms in the numerator and denominator, you could follow these steps:

i Multiplying algebraic fractions

To multiply two algebraic fractions together:

1. Factorize the numerator and denominator where possible.
2. Multiply the numerators together.
3. Multiply the denominators together.
4. Simplify, if possible, by cancelling any common factors in the numerator and denominator.

For more about cancelling common factors, see [Guide: Introduction to algebraic fractions](#).

It's important to see that you do not need to find a common denominator before multiplying.

i Example 6

Find $\frac{x}{2y} \cdot \frac{4y}{x^2}$.

No expressions in the numerator or denominator can be factorized. So you can proceed by multiplying the numerators and denominators together; doing this gives

$$\frac{x}{2y} \cdot \frac{4y}{x^2} = \frac{x \cdot 4y}{2y \cdot x^2} = \frac{4xy}{2x^2y}$$

Now you can simplify by cancelling any common factors to the numerator and denominator. There is a factor of 2 common to 4 and 2, and a factor of xy common to the numerator and the denominator. This means you can cancel both of these to get

$$\frac{4xy}{2x^2y} = \frac{4xy \div 2xy}{2x^2y \div 2xy} = \frac{2}{x} \quad \text{if } x \neq 0 \text{ and } y \neq 0$$

You don't need to write out the division terms in the cancellation step, as the next example demonstrates.

i Example 7

Find $\frac{x^2 - 9}{x^2 + 2x + 1} \cdot \frac{x + 1}{x - 3}$.

Start by factorizing numerators and denominators where possible. So here

$$x^2 - 9 = (x + 3)(x - 3)$$

$$x^2 + 2x + 1 = (x + 1)^2$$

Now substitute these into the problem, and multiply the numerators and denominators together.

$$\begin{aligned} \frac{x^2 - 9}{x^2 + 2x + 1} \cdot \frac{x + 1}{x - 3} &= \frac{(x + 3)(x - 3)}{(x + 1)^2} \cdot \frac{x + 1}{x - 3} \\ &= \frac{(x + 3)(x - 3)(x + 1)}{(x + 1)^2(x - 3)} \end{aligned}$$

Now you can simplify by cancelling some common factors. There are factors of $x - 3$ and $x + 1$ common to the numerator and the denominator, and cancelling these gives

$$\frac{(x + 3)(x - 3)(x + 1)}{(x + 1)^2(x - 3)} = \frac{x + 3}{x + 1} \quad \text{if } x \neq -1 \text{ and } x \neq 3$$

 Warning

Cancelling factors across the numerator and denominator only works when you have **one fraction at the end**. You cannot cancel in this way when you have more than one fraction.

Dividing algebraic fractions

As seen in [Guide: Arithmetic on numerical fractions](#), when you divide one fraction by another, you are asking “How many times does this fraction fit into the other?”

Similar to numerical fractions, a good way to handle this for algebraic fractions is to use the **reciprocal** of the second fraction. The reciprocal of a fraction is found by swapping its numerator and denominator.

For example, the reciprocal of $\frac{x}{3}$ is $\frac{3}{x}$ - provided that $x \neq 0$.

 Dividing algebraic fractions

To divide algebraic fractions, you can follow the **keep, change, flip** method:

1. **Keep** the first fraction the same.
2. **Change** division to multiplication.
3. **Flip** the second fraction (take its reciprocal).

This then turns the division problem into a multiplication problem.

i Example 8

Find $\frac{2y}{3} \div \frac{4}{y}$.

Keep the first fraction the same, change the division sign to a multiplication sign, and flip the second fraction so that $\frac{4}{y}$ becomes $\frac{y}{4}$. The problem then becomes:

$$\frac{2y}{3} \div \frac{4}{y} = \frac{2y}{3} \cdot \frac{y}{4} \quad \text{if } y \neq 0$$

Now you can follow the steps for a multiplication problem. So multiply the fractions together, and simplify by cancelling common factors to get:

$$\begin{aligned} \frac{2y}{3} \div \frac{4}{y} &= \frac{2y}{3} \cdot \frac{y}{4} \\ &= \frac{2y \cdot y}{3 \cdot 4} \\ &= \frac{2y^2}{12} = \frac{y^2}{6} \quad \text{if } y \neq 0 \end{aligned}$$

As with multiplication of algebraic fractions, after the keep, change, flip method you can factorize before the multiplication.

i Example 9

Find $\frac{x^2 - 9}{x^2 + 3x} \div \frac{x - 3}{x}$.

Keep the first fraction the same, change the division sign to a multiplication sign, and flip the second fraction to transform the problem into

$$\frac{x^2 - 9}{x^2 + 3x} \div \frac{x - 3}{x} = \frac{x^2 - 9}{x^2 + 3x} \cdot \frac{x}{x - 3}$$

You can then factorize the first of these to get

$$\frac{x^2 - 9}{x^2 + 3x} \cdot \frac{x}{x - 3} = \frac{(x - 3)(x + 3)}{x(x + 3)} \cdot \frac{x}{x - 3}$$

Now multiply the fractions together, and simplify by cancelling common factors.

$$\begin{aligned} \frac{(x + 3)(x - 3)}{x(x + 3)} \cdot \frac{x}{x - 3} &= \frac{(x + 3)(x - 3)x}{x(x + 3)(x - 3)} \\ &= 1 \quad \text{if } x \neq 0, -3, 3 \end{aligned}$$

Here, the restrictions $x \neq 0$ and $x \neq -3$ come from the denominator $x(x + 3)$, and $x \neq 3$ comes from the denominator $x - 3$ in the fraction you are multiplying by.

Compound algebraic fractions

Sometimes, fractions can have other fractions inside them. These are called **compound fractions**. Here are some examples

$$\frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x}} \qquad \frac{\frac{x}{y}}{\frac{a}{b}} \qquad \frac{1}{1 + \frac{1}{c}}$$

It is almost always better to rewrite these fractions in a more standard fractional form, with no fractions in either the numerator or the denominator. The goal is always the same; you need to simplify them so that there is only one single fraction left.

A fraction A/B is exactly the same as saying $A \div B$. So every compound fraction can be rephrased as a division problem, which you know how to do!

i Simplifying compound algebraic fractions

To simplifying a compound algebraic fraction, you can rewrite the large fraction as a division problem and use the keep, change, flip rule.

1. Remember that “a fraction over a fraction” means division, and rewrite the compound fraction as a division problem.
2. Apply the keep, change, flip rule.
3. Multiply the fractions together.
4. Factorize and simplify, if possible.

This also clears the denominators of the smaller fractions and leaves an ordinary fraction.

This method is best illustrated using some examples.

i Example 10

Simplify $\frac{\frac{2a}{3}}{\frac{a+1}{6}}$.

First of all, rewrite the compound algebraic fraction as a division problem. The compound fraction

$$\frac{\frac{2a}{3}}{\frac{a+1}{6}} = \frac{2a}{3} \div \frac{a+1}{6}$$

Now, you can apply the keep, change, flip method, changing the division problem into a multiplication one, getting:

$$\frac{2a}{3} \div \frac{a+1}{6} = \frac{2a}{3} \cdot \frac{6}{a+1}$$

Multiplying the fractions together, and simplifying gives:

$$\begin{aligned} \frac{2a}{3} \cdot \frac{6}{a+1} &= \frac{2a \cdot 6}{3(a+1)} \\ &= \frac{12a}{3(a+1)} \\ &= \frac{4a}{a+1} \quad \text{if } a \neq -1 \end{aligned}$$

The final example in this guide requires all the concepts seen so far, and also illustrates the importance of brackets in evaluating compound fractions.

i Example 11

Simplify $\frac{\frac{1}{x} + \frac{2}{y^2}}{\frac{3}{xy}}$.

Here, you can rewrite the compound fraction as a division problem. You can notice there are two fractions in the numerator, both of which must be divided by what's in the denominator. You can ensure this by using brackets to say that

$$\frac{\frac{1}{x} + \frac{2}{y^2}}{\frac{3}{xy}} = \left(\frac{1}{x} + \frac{2}{y^2} \right) \div \frac{3}{xy}$$

The order of operations (see [Guide: Order of operations in algebra]) suggests that you should deal with the bracketed term first. Using cross-multiplication to do this gives

$$\frac{1}{x} + \frac{2}{y^2} = \frac{y^2}{xy^2} + \frac{2x}{xy^2} = \frac{2x + y^2}{xy^2}$$

and putting into the problem gives

$$\left(\frac{1}{x} + \frac{2}{y^2} \right) \div \frac{3}{xy} = \frac{2x + y^2}{xy^2} \div \frac{3}{xy}$$

Now you have only two fractions, you can use the keep, change, flip method to get

$$\frac{2x + y^2}{xy^2} \div \frac{3}{xy} = \frac{2x + y^2}{xy^2} \cdot \frac{xy}{3}$$

Finally you can multiply together and cancel to get:

$$\frac{2x + y^2}{xy^2} \cdot \frac{xy}{3} = \frac{xy(2x + y^2)}{3xy^2} = \frac{2x + y^2}{3y} \quad \text{if } x \neq 0 \text{ and } y \neq 0.$$

Therefore, your final answer is

$$\frac{\frac{1}{x} + \frac{2}{y^2}}{\frac{3}{xy}} = \frac{2x + y^2}{3y} \quad \text{if } x \neq 0 \text{ and } y \neq 0.$$

Quick check problems

1. Calculate $\frac{2}{x} + \frac{3}{x}$.

2. Calculate $\frac{1}{x} + \frac{1}{2x}$. (a) $\frac{1}{3x}$ (b) $\frac{2}{3x}$ (c) $\frac{3}{2x}$ (d) $\frac{1}{2x}$ (e) $\frac{x+1}{2x}$

3. Calculate $\frac{1}{x} - \frac{3}{x^2}$.

4. Calculate $\frac{2}{x} \cdot \frac{x}{5}$. (a) $\frac{x}{10}$ (b) $\frac{2x}{5}$ (c) $\frac{x}{5}$ (d) $\frac{10}{x}$ (e) $\frac{2}{5}$

5. Calculate $\frac{3x}{4} \div \frac{x}{2}$.

6. Simplify $\frac{\frac{2x}{3}}{\frac{x}{6}}$. (a) 2 (b) 4 (c) $\frac{x}{2}$ (d) $\frac{6}{x}$ (e) $4x$

Further reading

For more questions on the subject, please go to [Questions: Arithmetic on algebraic fractions](#).

Version history

v1.0: initial version created 12/25 by Donald Campbell as part of a University of St Andrews VIP project.

[This work is licensed under CC BY-NC-SA 4.0.](#)