

Proof: Law of total probability and Bayes' theorem

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Summary

This proof sheet demonstrates that the law of total probability and Bayes' theorem are true.

Before reading this proof sheet, it is recommended that you read [Guide: Conditional probability](#) and [Guide: Law of total probability and Bayes' theorem](#).

Proof of the law of total probability

First of all, here is a restatement of the law of total probability from [Guide: Law of total probability and Bayes' theorem](#):

i Definition of the law of total probability

Suppose an event B depends on several possible scenarios. These scenarios can be described by events A_1, A_2, \dots, A_n , that are:

- **Mutually exclusive:** they cannot occur at the same time, and
- **Exhaustive:** one of them must always occur.

Then, the **law of total probability** states that the probability of event B is:

$$\mathbb{P}(B) = \sum_{i=1}^n \mathbb{P}(A_i) \mathbb{P}(B | A_i)$$

The proof of the law of total probability comes directly from the definition of conditional probability given in [Guide: Conditional probability](#):

$$\mathbb{P}(B | A_i) = \frac{\mathbb{P}(B \cap A_i)}{\mathbb{P}(A_i)}$$

Multiplying by $\mathbb{P}(A_i)$ gives the multiplication rule (again from [Guide: Conditional probability](#))

:

$$\mathbb{P}(B \cap A_i) = \mathbb{P}(B | A_i)\mathbb{P}(A_i)$$

As scenarios A_1, A_2, \dots, A_n are mutually exclusive (so $A_i \cap A_j = \emptyset$ for all $1 \leq i \neq j \leq n$) and exhaustive ($\bigcup_{1 \leq i \leq n} A_i = B$), it follows from results in set theory (see [Guide: Operations on sets]) that:

$$\mathbb{P}(B) = \mathbb{P}(B \cap A_1) + \mathbb{P}(B \cap A_2) + \dots + \mathbb{P}(B \cap A_n) = \sum_{i=1}^n \mathbb{P}(B \cap A_i)$$

Substituting the above expressions gives:

$$\mathbb{P}(B) = \mathbb{P}(B | A_1)\mathbb{P}(A_1) + \mathbb{P}(B | A_2)\mathbb{P}(A_2) + \dots + \mathbb{P}(B | A_n)\mathbb{P}(A_n) = \sum_{i=1}^n \mathbb{P}(B | A_i)\mathbb{P}(A_i)$$

Which results in the law of total probability:

$$\mathbb{P}(B) = \sum_{i=1}^n \mathbb{P}(A_i)\mathbb{P}(B | A_i)$$

Proof of Bayes' theorem

Here is the statement of Bayes' theorem from [Guide: Law of total probability and Bayes' theorem](#):

i Statement of Bayes' Theorem

If A and B are events with $\mathbb{P}(B) > 0$, then Bayes' Theorem states:

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(B | A)\mathbb{P}(A)}{\mathbb{P}(B)}.$$

where:

- $\mathbb{P}(A | B)$ is the probability of A given B ,
- $\mathbb{P}(B | A)$ is the probability of B given A ,
- $\mathbb{P}(A)$ and $\mathbb{P}(B)$ are the individual probabilities of A and B , respectively.

Bayes' Theorem is derived directly from the definition of conditional probability: see [Guide: Conditional probability](#). Start with the conditional probabilities of two events A and B :

$$(1) \quad \mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \quad \text{and} \quad (2) \quad \mathbb{P}(B | A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

You can rearrange (2) by multiplying both sides by $\mathbb{P}(A)$, giving the multiplication rule:

$$\mathbb{P}(A \cap B) = \mathbb{P}(B | A)\mathbb{P}(A)$$

Substitute this result into equation (1) to get:

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(B | A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

This gives Bayes' Theorem, a way to reverse conditional probabilities when direct calculation is difficult.

Further reading

[Click this link to go back to Guide: Law of total probability and Bayes' theorem.](#)

Version history

v1.0: initial version created 04/25 by Sophie Chowgule as part of a University of St Andrews VIP project.

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